Chapter 5. Constitutive Models

This chapter describes the theoretical background behind the constitutive models that are available in FEBio. Most materials are derived from a hyperelastic strain-energy function. Please consult section 2.4 for more background information on this type of material.

5.1. Linear Elasticity

In the theory of linear elasticity the Cauchy stress tensor is a linear function of the small strain tensor $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\sigma} = \boldsymbol{\mathcal{C}} : \boldsymbol{\varepsilon} . \tag{5.1}$$

Here, $\boldsymbol{\mathcal{C}}$ is the fourth-order elasticity tensor that contains the material properties. In the most general case this tensor has 21 independent parameters. However, in the presence of material symmetry the number of independent parameters is greatly reduced. For example, in the case of isotropic linear elasticity only two independent parameters remain. In this case, the elasticity tensor is given by

$$\mathcal{C}_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) . \tag{5.2}$$

The material coefficients λ and μ are known as the Lamé parameters. Using this equation, the stress-strain relationship can be written as

$$\sigma_{ij} = \lambda \left(\operatorname{tr} \boldsymbol{\varepsilon} \right)^2 \delta_{ij} + 2\mu \varepsilon_{ij} . \tag{5.3}$$

If the stress and strain are represented as Voigt vectors, the constitutive equation can be rewritten in matrix form as

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{pmatrix} = \begin{pmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \lambda_{12} \\ \lambda_{23} \\ \lambda_{13} \end{pmatrix} . \tag{5.4}$$

The strain measures λ_{ij} are the *engineering strains* and are given by $\lambda_{ij} = 2\varepsilon_{ij}$.

The following table relates the Lame parameters to the more familiar Young's modulus *E* and Poisson's ratio ν or to the bulk modulus *K* and shear modulus *G*.

	E,ν	λ,μ	K,G
E,ν		$E=\frac{\mu}{\lambda+\mu}(2\mu+3\lambda)$ $\nu=\frac{\lambda}{2(\lambda+\mu)}$	$E=\frac{9KG}{3K+G}$ $\nu=\frac{3K-2G}{6K+2G}$
λ,μ	$\lambda=\frac{\nu E}{(1+\nu)(1-2\nu)}$ $\mu=\frac{E}{2(1+\nu)}$		$\lambda=K-\frac{2}{3}G$ $\mu=G$
K,G	$K=\frac{E}{3(1-2\nu)}$ $G=\frac{E}{2(1+\nu)}$	$K=\lambda+\frac{2}{3}\mu$ $G=\mu$	

It is of some interest to note that the theoretical range of the Poisson's ratio for an isotropic material is $-1\le\nu\le 0.5$. Materials with Poisson's ratio (close to) 0.5 are known as (nearly-) incompressible materials. For these materials, the bulk modulus approaches infinity. Most materials have a positive Poisson's ratio, although there do exist some materials with a negative ratio. These materials are known as *auxetic* materials and they have the remarkable property that they expand under tension.

The linear stress-strain relationship can also be derived from a strain-energy function such as in the case of hyperelastic materials. In this case the linear strain-energy is given by

$$W=\frac{1}{2}\boldsymbol{\varepsilon}:\mathbf{C}\boldsymbol{\varepsilon}\,. \tag{5.5}$$

The stress is then similarly derived from $\boldsymbol{\sigma}=\frac{\partial W}{\partial\boldsymbol{\varepsilon}}$. In the case of isotropic elasticity, (5.5) can be simplified:

$$W=\frac{1}{2}\lambda\left(\operatorname{tr}\boldsymbol{\varepsilon}\right)^2+\mu\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon}\,. \tag{5.6}$$

The Cauchy stress is now given by

$$\boldsymbol{\sigma}=\lambda\left(\operatorname{tr}\boldsymbol{\varepsilon}\right)\mathbf{1}+2\mu\boldsymbol{\varepsilon}\,. \tag{5.7}$$

5.2. Isotropic Elasticity

The linear elastic material model as described in section 5.1 is only valid for small strains and small rotations. A first modification to this model to the range of nonlinear deformations is given by the St. Venant-Kirchhoff model [1], which in FEBio is referred to as *isotropic elasticity*. This model is objective for large strains and rotations. For the isotropic case it can be derived from the following hyperelastic strain-energy function:

$$W = \frac{1}{2}\lambda\left(\mathrm{tr}\,\mathbf{E}\right)^2 + \mu\mathbf{E} : \mathbf{E}\,. \tag{5.8}$$

The second Piola-Kirchhoff stress can be derived from this:

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{E}} = \lambda\left(\mathrm{tr}\,\mathbf{E}\right)\mathbf{1} + 2\mu\mathbf{E}\,. \tag{5.9}$$

Note that these equations are similar to the corresponding equations in the linear elastic case, only the small strain tensor is replaced by the Lagrangian elasticity tensor **E**.

5.3. Neo-Hookean Hyperelasticity

This is a compressible neo-Hookean material. It is derived from the following hyperelastic strain energy function [1]:

$$W = \frac{\mu}{2}\left(I_1 - 3\right) - \mu\ln J + \frac{\lambda}{2}\left(\ln J\right)^2.$$

The parameters μ and λ are the Lamé parameters from linear elasticity. This model reduces to the isotropic linear elastic model for small strains and rotations.

The neo-Hookean material is an extension of Hooke's law for the case of large deformations. It is useable for plastics and rubber-like substances. A generalization of this model is the Mooney-Rivlin material, which is often used to describe the elastic response of biological tissue.

In FEBio this constitutive model uses a standard displacement-based element formulation and a "coupled" strain energy, so care must be taken when modeling materials with nearly-incompressible material behavior to avoid element locking.

5.4. Mooney-Rivlin Hyperelasticity

This material model is a hyperelastic Mooney-Rivlin type with uncoupled deviatoric and volumetric behavior. The uncoupled strain energy W is given by:

$$W = C_1\left(\tilde{I}_1 - 3\right) + C_2\left(\tilde{I}_2 - 3\right) + \frac{1}{2}K\left(\ln J\right)^2.$$

C_1 and C_2 are the Mooney-Rivlin material coefficients, $\tilde{I}_1$ and $\tilde{I}_2$ are the invariants of the deviatoric part of the right Cauchy-Green deformation tensor, $\tilde{\boldsymbol{C}} = \tilde{\boldsymbol{F}}^T\tilde{\boldsymbol{F}}$, where $\tilde{\boldsymbol{F}} = J^{(-1/3)}\boldsymbol{F}$, $\boldsymbol{F}$ is the deformation gradient and $J = \det\left(\boldsymbol{F}\right)$ is the Jacobian of the deformation. When $C_2 = 0$, this model reduces to an uncoupled version of the