Porous Law for snow and firn in Elmer/Ice

Olivier Gagliardini

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We present here the equations of the Snow/Firn rheological law implemented in Elmer/Ice (`PorousSolve`) and analytical solutions for purpose of testing the implementation. This law is a compressible Norton-Hoff type law, which response depends on the relative density D of the material. At the limit, when $D = 1$, the Snow/Firn law reduces exactly to the classical Glen's flow law. This law was initialy proposed by Duva and Crow (1994) and adapted to the snow and firn by Gagliardini and Meyssonnier (1997). This law has been used later for different applications conducted with different models (Lüthi and Funk, 2000, 2001; Boutillier, 2004; Zwinger et al., 2007).

Snow/Firn law

To formulate the law, let first decompose the Cauchy stress tensor $\boldsymbol{\sigma}$ and the strain-rate tensor $\dot{\boldsymbol{\epsilon}}$ in their deviatoric parts $\boldsymbol{\tau}$ and $\dot{\boldsymbol{e}}$, and isotropic parts $p\boldsymbol{I}$ and $\dot{\epsilon}_m \boldsymbol{I}/3$, respectively:

$$\boldsymbol{\tau} = \boldsymbol{\sigma} - \frac{\mathrm{tr}\,\boldsymbol{\sigma}}{3}\boldsymbol{I} = \boldsymbol{\sigma} + p\boldsymbol{I}\,, \tag{1}$$

and

$$\dot{\boldsymbol{e}} = \dot{\boldsymbol{\epsilon}} - \frac{\mathrm{tr}\,\dot{\boldsymbol{\epsilon}}}{3}\boldsymbol{I} = \dot{\boldsymbol{\epsilon}} - \frac{1}{3}\dot{\epsilon}_m\boldsymbol{I}\,. \tag{2}$$

We here adopt the same definition for the isotropic pressure as in Elmer ($p = -\,\mathrm{tr}\,\boldsymbol{\sigma}/3$). With these notations, compression is negative for stress but positive for the isotropic pressure. $\dot{\epsilon}_m = \mathrm{d}V/V$ is the relative change of volume. For an incompressible fluid (ice), $\dot{\epsilon}_m = 0$.

Let's now define invariants for the strain-rate:

$$\gamma_e^2 = 2\,\mathrm{tr}\,(\dot{\boldsymbol{e}})^2 = 2\dot{e}_{ij}\dot{e}_{ij}\,, \quad \dot{\epsilon}_D^2 = \frac{\gamma_e^2}{a} + \frac{\dot{\epsilon}_m^2}{b}\,, \tag{3}$$

and for the stress:

$$\tau^2 = \frac{1}{2}\,\mathrm{tr}\,\boldsymbol{\tau}^2 = \frac{1}{2}\tau_{ij}\tau_{ij}\,, \quad \sigma_D^2 = a\tau^2 + bp^2\,. \tag{4}$$

The two parameters $a = a(D)$ et $b = b(D)$ are only function of the relative density D, which is then needed as an internal variable for the Snow/Firn law. The form of these two functions is described below.

The rheological law expresses the relation between the deviatoric and isotropic parts of the stress and that ones of the strain-rate, such as:

$$\boldsymbol{\tau} = \frac{2}{a} B_n^{-1/n} \dot{\epsilon}_D^{(1-n)/n} \dot{\boldsymbol{e}} \,, \tag{5}$$

and

$$p = -\frac{1}{b} B_n^{-1/n} \dot{\epsilon}_D^{(1-n)/n} \dot{\epsilon}_m \,. \tag{6}$$

Inverse relations can be obtained by noticing the relation between stress and strain-rate invariants, which writes:

$$\dot{\epsilon}_D = B_n \sigma_D^n \,. \tag{7}$$

Using this relation between invariants, one obtains:

$$\dot{\boldsymbol{e}} = \frac{a}{2} B_n \sigma_D^{n-1} \boldsymbol{\tau} \,, \tag{8}$$

and

$$\dot{\epsilon}_m = -b B_n \sigma_D^{n-1} p \,. \tag{9}$$

The parameters a and b are function of the relative density $D = \rho/\rho_i$, where ρ is the snow density and ρ_i is the ice density. In the limit case where $D = 1$ (ice), $a = 1$ and $b = 0$, and the Snow/Firn law reduces to the classical incompressible Glen's flow law:

$$\boldsymbol{\tau} = 2 B_n^{-1/n} \gamma_e^{(1-n)/n} \dot{\boldsymbol{\epsilon}} \,, \text{ or in its inverse form } \dot{\boldsymbol{\epsilon}} = \frac{B_n}{2} \tau^{n-1} \boldsymbol{\tau} \tag{10}$$

(Note: $B_n = 2A = \eta_0^{-n}$, where A is the fludity parameter used in the ISMIP tests and η_0 is the viscosity as defined in Elmer for the Navier-Stokes solver).
For large relative density ($0.81 < D \leq 1$), one can adopt the proposed analytical solution for the two functions a and b (Duva and Crow, 1994):

$$a_0(D) = \frac{1 + 2(1-D)/3}{D^{2n/(n+1)}} \,, \text{ et } \quad b_0(D) = \frac{3}{4} \left(\frac{(1-D)^{1/n}}{n[1 - (1-D)^{1/n}]} \right)^{2n/(n+1)} \,. \tag{11}$$

For smaller relative density, a parametrized form of the two functions a and b is adopted, by fitting cold room experiments and densification measured at Site 2. Under uniaxial compression, the section of the sample cannot decrease, implying that $3a > 2b$. We finally adopt the following expression for the two functions a and b (see Figure 1):

$$\begin{aligned}
a(D) &= \begin{cases} \mathrm{e}^{(13.22240 - 15.78652\, D)} & 0.4 \leq D \leq 0.81 \,, \\ a_0(D) & 0.81 < D \leq 1 \,, \end{cases} \\
b(D) &= \begin{cases} \mathrm{e}^{(15.09371 - 20.46489\, D)} & 0.4 \leq D \leq 0.81 \,, \\ b_0(D) & 0.81 < D \leq 1 \,. \end{cases}
\end{aligned} \tag{12}$$

Various expressions have been used in the different applications, by adopting slightly different numbers in the parametrization for small relative density.

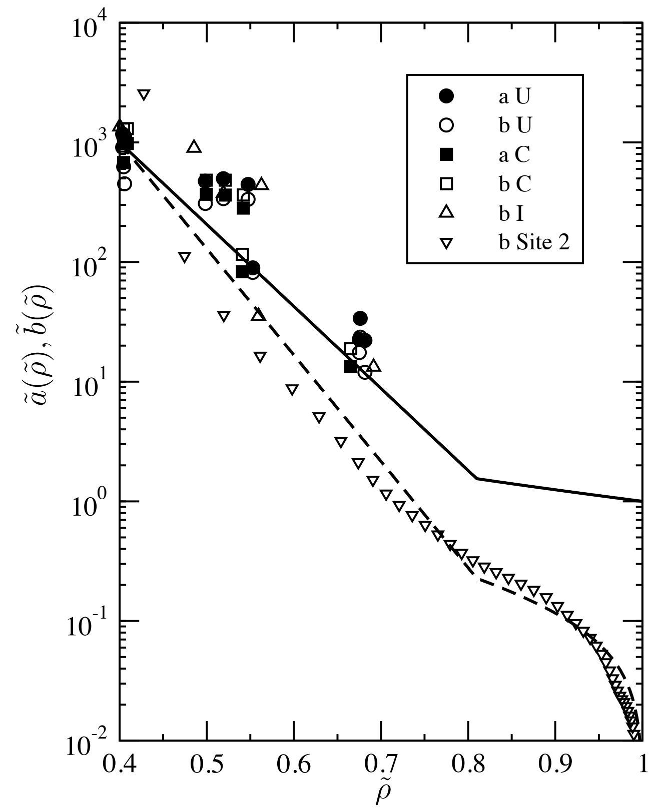

Figure 1: Evolution of the parameters a (line) and b (dashed) as a function of the relative density D. Mechanical tests from Landauer (1957) and densification at Site 2 are indicated by the symbols (U uni-axial tests, C confined uni-axial test and I isotropic compression) (adapted from Gagliardini and Meyssonnier, 1997).

Analytical Solution

The corresponding problems solved with Elmer can be downloaded on the svn (in `Test/Test_Porous.tar.gz`). An excel file is also available to get analytical value for other input parameter values (in `DocSolverLGGE/AnalyticalSolutionPorous.xls`).

All the solutions are given for a box of size $L \times l \times h$. Numerical solution are given for the following numerical values:

n	3	
B_n	20	MPa^{-3} a^{-1}
ρ_i	900	kg m^{-3}
g	9.81	m s^{-2}
D	0.5	
a	206.2605	
b	129.1875	

Table 1: Numerical values of the parameters

In all the following examples, the common boundary conditions are $u(x=0) = v(y=0) = w(z=0) = 0$. Other boundary conditions are specified for each particular examples.

Uniaxial stress-driven compression test

The supplementary boundary condition for this example is $\sigma_{zz}(z=h) = \bar{\sigma} = -0.01$MPa.

The stress components are then $\sigma_{zz} = \bar{\sigma}$, and all other components are nulls, so that $p = -\bar{\sigma}/3$. The deviatoric stress components are therefore $\tau_{zz} = 2\bar{\sigma}/3$ and $\tau_{xx} = \tau_{yy} = -\bar{\sigma}/3$. From Equation (4), the invariants are found to be $\tau^2 = \bar{\sigma}^2/3$ and $\sigma_D^2 = (a/3 + b/9)\bar{\sigma}^2$.

From Equations (8) and (9), one can derive the non-null strain-rate components:

$$\dot{\epsilon}_{zz} = B_n \left(\frac{a}{3} + \frac{b}{9} \right)^{(n+1)/2} \bar{\sigma}^n \,, \tag{13}$$

and

$$\dot{\epsilon}_{xx} = \dot{\epsilon}_{yy} = \frac{B_n}{18} \left(\frac{a}{3} + \frac{b}{9} \right)^{(n-1)/2} (2b - 3a)\bar{\sigma}^n \,. \tag{14}$$

The numerical solution is then $\dot{\epsilon}_{zz} = -0.1381$ a^{-1} and $\dot{\epsilon}_{xx} = \dot{\epsilon}_{yy} = 0.03328$ a^{-1}.

Uniaxial velocity-driven compression test

The supplementary boundary condition for this example is $w(z=h) = \bar{w} = -0.01$ m/a.

The vertical strain-rate is $\dot{\epsilon}_{zz} = \bar{w}/h$. The only non-zero stress components is then σ_{zz}, so that $p = -\sigma_{zz}/3$. The deviatoric stress components are therefore $\tau_{zz} = 2\sigma_{zz}/3$ and $\tau_{xx} = \tau_{yy} = -\sigma_{zz}/3$. From Equation (4), the invariants are found to be $\tau^2 = \sigma_{zz}^2/3$ and $\sigma_D^2 = (a/3 + b/9)\sigma_{zz}^2$.

From Equations (8) and (9), one can obtain the vertical Cauchy stress:

$$\sigma_{zz} = B_n^{-1} \left(\frac{a}{3} + \frac{b}{9} \right)^{-(n+1)/(2n)} \left(\frac{\bar{w}}{h} \right)^{1/n} \,, \tag{15}$$

and the transverse strain-rate components:

$$\dot{\epsilon}_{xx} = \dot{\epsilon}_{yy} = \frac{2b - 3a}{2b + 6a} \frac{\bar{w}}{h} \,. \tag{16}$$

Isotropic compression test

The supplementary boundary condition for this example is $\sigma_{xx}(x=L) = \sigma_{yy}(y=l) = \sigma_{zz}(z=h) = \bar{\sigma} = -0.01$MPa.

The stress components are then $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \bar{\sigma}$, and all other components are nulls, so that $p = -\bar{\sigma}$. The deviatoric stress components are all nulls. From Equation (4), the invariant is found to be $\sigma_D^2 = b\bar{\sigma}^2$.

From Equation (9), one can derive the non-null strain-rate components:

$$\dot{\epsilon}_{xx} = \dot{\epsilon}_{yy} = \dot{\epsilon}_{zz} = \frac{\dot{\epsilon}_m}{3} = \frac{b^{(n+1)/2}}{3} B_n \bar{\sigma}^n \,. \tag{17}$$

The numerical solution is then $\dot{\epsilon}_{xx} = \dot{\epsilon}_{yy} = \dot{\epsilon}_{zz} = -0.1113$ a^{-1}.

Uniaxial stress-driven confined compression test

The supplementary boundary conditions for this example are $\sigma_{zz}(z = h) = \bar{\sigma} = -0.01$MPa and $u(x = L) = v(y = l) = 0$.

The vertical stress component is then $\sigma_{zz} = \bar{\sigma}$. From the lateral boundary conditions, we have $\dot{\epsilon}_{xx} = \dot{\epsilon}_{yy} = 0$ so that $\dot{e}_{xx} = \dot{e}_{yy} = -\dot{\epsilon}_{zz}/3$ and $\dot{e}_{zz} = 2\dot{\epsilon}_{zz}/3$. Strain-rate invariants reduce to $\dot{\epsilon}_m = \dot{\epsilon}_{zz}$, $\gamma_e^2 = 4\dot{\epsilon}_{zz}/3$ and $\dot{\epsilon}_D^2 = (4/(3a) + 1/b)\dot{\epsilon}_{zz}^2$.

One can then estimate τ_{zz} and p:

$$\tau_{zz} = \frac{4}{3a} B_n^{-1/n} \left(\frac{4}{3a} + \frac{1}{b}\right)^{(1-n)/(2n)} \dot{\epsilon}_{zz}^{1/n} \,, \text{ and } p = -\frac{1}{b} B_n^{-1/n} \left(\frac{4}{3a} + \frac{1}{b}\right)^{(1-n)/(2n)} \dot{\epsilon}_{zz}^{1/n} \,, \tag{18}$$

so that

$$\sigma_{zz} = \bar{\sigma} = B_n^{-1/n} \left(\frac{4}{3a} + \frac{1}{b}\right)^{(1+n)/(2n)} \dot{\epsilon}_{zz}^{1/n} \,. \tag{19}$$

Inversion of the previous equation allows to infer the strain-rate and the missing deviatoric stress components:

$$\dot{\epsilon}_{zz} = B_n \left(\frac{4}{3a} + \frac{1}{b}\right)^{-(n+1)/2} \bar{\sigma}^n \,, \tag{20}$$

and

$$\tau_{xx} = \tau_{yy} = -\frac{2}{3a} \left(\frac{4}{3a} + \frac{1}{b}\right)^{-1} \bar{\sigma} \,. \tag{21}$$

The numerical solution is then $\dot{\epsilon}_{zz} = -0.0991$ a^{-1}, $\tau_{xx} = \tau_{yy} = 0.002275$ MPa, $\tau_{zz} = -0.00455$ MPa and $p = 0.00545$ MPa.

Uniaxial compression of a sample under gravity

This solution is for an infinitely large media of relative density D flowing under gravity. The supplementary boundary conditions for this example are $u(x = L) = v(y = l) = 0$ and the upper surface $z = h$ is stress-free .

The solution is similar to the previous one, excepted that $\sigma_{zz} = \rho_i D g(z - h)$, $\sigma_{xx} = \sigma_{xx}(y, z)$ and $\sigma_{yy} = \sigma_{yy}(x, z)$. From the lateral boundary conditions, we have $\dot{\epsilon}_{xx} = \dot{\epsilon}_{yy} = 0$ so that $\dot{e}_{xx} = \dot{e}_{yy} = -\dot{\epsilon}_{zz}/3$ and $\dot{e}_{zz} = 2\dot{\epsilon}_{zz}/3$. Strain-rate invariants reduce to $\dot{\epsilon}_m = \dot{\epsilon}_{zz}$, $\gamma_e^2 = 4\dot{\epsilon}_{zz}/3$ and

$\dot{\epsilon}_D^2 = (4/(3a) + 1/b)\dot{\epsilon}_{zz}^2$.

The solution is

$$\dot{\epsilon}_{zz} = \dot{\epsilon}_m = B_n \left(\frac{4}{3a} + \frac{1}{b} \right)^{-(n+1)/2} (\rho_i D g (z-h))^n \,, \tag{22}$$

and

$$p = -\left(1 + \frac{4b}{3a}\right)^{-1} \rho_i D g (z-h)\,, \text{ and } \tau_{zz} = \left[1 - \left(1 + \frac{4b}{3a}\right)^{-1}\right] \rho_i D g (z-h)\,, \tag{23}$$

and

$$\tau_{xx} = \tau_{yy} = -\left(2 + \frac{3a}{2b}\right)^{-1} \rho_i D g (z-h)\,. \tag{24}$$

Integration of the strain-rate allows to evaluate the velocities:

$$w(z) = B_n \left(\frac{4}{3a} + \frac{1}{b} \right)^{-(n+1)/2} (\rho_i D g)^n \frac{(z-h)^{n+1} - h^{n+1}}{n+1}\,, \text{ and } u = v = 0\,. \tag{25}$$

Inferring the function a and b from borehole measurements

For this inversion, one needs to know the density evolution with depth $\rho = \rho(z)$, the surface accumulation a_c [m weq a^{-1}] and the temrature profile $T = T(z)$. Assuming that the density evolution is solely due to vertical compaction (infinitely large media case), the vertical strain-rate is given by (Pimienta, 1987):

$$\dot{\epsilon}_{zz}(z) = \frac{a_c}{D^2 \rho_i} \frac{\partial D}{\partial z} \,. \tag{26}$$

For a uniaxial confined compression, the relation between the hydrostatic pressure (vertical stress) and the vertical strain-rate is given by Equation (20). The hydrostatic pressure is computed as the load of snow over the layer of altitude z, so that:

$$\sigma_{zz}(z) = \int_z^{z_s} \rho_i D g \, \mathrm{d}z \,. \tag{27}$$

And the two parameters can be estimated as:

$$\left(\frac{4}{3a} + \frac{1}{b} \right) = \frac{\sigma_{zz}^{2n/(n+1)} B_n^{2/(n+1)}}{\dot{\epsilon}_{zz}^{2/(n+1)}} \,, \tag{28}$$

where $\sigma_{zz}(z)$, $\dot{\epsilon}_{zz}(z)$ and $B_n(T(z))$ are evaluated at each depth. The problem in that form is ill-posed because we have two functions to be determined and only one equation. One can adopt that the ratio $a/b = a_0/b_0$, where a_0 and b_0 are defined by (11). One should also remember that $3a > 2b$ and include this constraint in $a/b = a_0/b_0$.

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